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**NEW YORK UNIVERSITY**
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

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# A Plane Wave Expansion Theorem for Cylindrically Radiated Fields

S. N. KARP and N. R. ZITRON

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Errata
EM-199

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Page 5, Equation 2(a) $(\beta^{\lambda}(0)$ should read $(\beta^{\lambda}(0))$

Page 7, line 13 Equation should read:

$$g^{(m-p)} = \sum_{\lambda=0}^{m-p} \sum_{\ell=0}^{\lambda} (-1)^{\ell} \frac{c_{\ell}^{\lambda}}{\lambda!} \beta^{\ell} (\beta^{\lambda-\ell})^{(m-p)} g[\lambda]$$

Page 8 Then $I_{2t}$= ..... should read:

$$I_{2t} = \begin{cases} 0 & m \text{ odd} \\ \dfrac{\Gamma(t + 1/2)}{(kd)^{\frac{2t+1}{2}}} & m \text{ even} \quad t = 0,1,2,\ldots \end{cases} .$$

Page 9, Footnote should read:

It should be pointed out here that the definition of the complex scattering amplitude of the far field $f(\beta)$ in this paper differs from that of the previous paper [1]. Let $f$ denote the complex scattering amplitude of the far field in this paper and $\tilde{f}$ denote the one in the previous paper. The relation between the two amplitudes is then

$$f = \sqrt{\frac{k}{2\pi}}\, e^{i\frac{\pi}{4}}\, \tilde{f} .$$

Page 10, line 7 Equation should read:

$$u(r,\theta) \sim \sqrt{\frac{2}{\pi kr}}\, e^{i(kr - \frac{\pi}{4})}\, [\hat{f}(\theta)]$$

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# A PLANE WAVE EXPANSION THEOREM FOR CYLINDRICALLY RADIATED FIELDS

S. N. Karp and N. R. Zitron*

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## Abstract

An asymptotic expansion for two-dimensional outwardly radiating fields is developed from an integral representation of these fields by means of a saddle point integration. The expansion is given in terms of inverse powers of the distance from a point in a fixed region A to a point in a circular neighborhood at a large distance from A. The coefficients are expressed in terms of plane waves and linear combinations of derivatives of plane waves with respect to angle of incidence. The theorem may be employed in scattering problems in reducing scattering of arbitary two-dimensional fields by arbitrary cylinders to scattering of plane waves by the arbitrary cylinders.

Introduction

The radiation from a collection of cylindrical sources within a bounded cylindrical domain, when viewed by an observer at an infinite distance, cannot be distinguished from a plane wave. As the observer approaches closer to the source region, the nature of the radiation may appear cylindrical or perhaps even more complicated. If, however, the distance from the source is sufficiently large, it may be said that the wave is almost, but not quite, a plane wave.

This paper is concerned with the relation between a general cylindrically radiated wave and a plane wave. It is shown here that the relation is expressible as an asymptotic series involving plane waves and their derivatives of various orders with respect to angle. A few terms of this series, obtained previously by Zitron and Karp [1], were utilized to good advantage in dealing with multiple scattering of plane waves by two widely spaced parallel cylinders of arbitrary shape. The present paper gives an expression for the general term of the series for which the coefficient is a linear combination of plane waves and their derivatives with respect to angle of incidence. This representation also provides a more efficient way of calculating the terms of the expansion than was used before [1]. These results were reported earlier in abstracted form [2], [3].

The major result presented in this paper is an asymptotic expansion, in a distant finite region of an integral representation for a cylindrically radiated field. Application of the saddle point method to this integral yields a power series in inverse half-integral powers of kd, where k is the propagation

constant and d is a long distance by which the origin O of a coordinate system located in the source region has been translated. Since the coefficients in the asymptotic expansion turn out to be linear combinations of plane waves and their derivatives of various orders with respect to angle of incidence, the expansion can be used to represent the given field in a new coordinate system as a plane wave plus a correction for the curvature of the wavefront in the neighborhood of a scatterer located near the origin d of the new coordinate system. The correction terms appear in the form of derivatives of plane waves with respect to angle of incidence. The latter is merely a parameter. Thus the theorem has special utility in scattering problems, since the response to the incident wave can be given in terms of derivatives of the response to a plane wave as was done by the authors [1] in a previous paper. It is possible, then, by means of this expansion, to determine the scattered field when radiation from a cylindrical region is incident upon a cylindrical scatterer and also to reduce multiple scattering problems to single scattering problems. Both of these cases may be treated in a quite general manner without reference to the detailed nature of the incident radiation or the shape or composition of the scatterer. In this respect, the expansion theorem may be used in place of addition theorems which are applicable only when suited to the special geometry of the sources and scatterers and which, even in such cases, may be clumsier to apply.

Derivation of the Expansion

The geometry of the situation is illustrated in Figure 1.

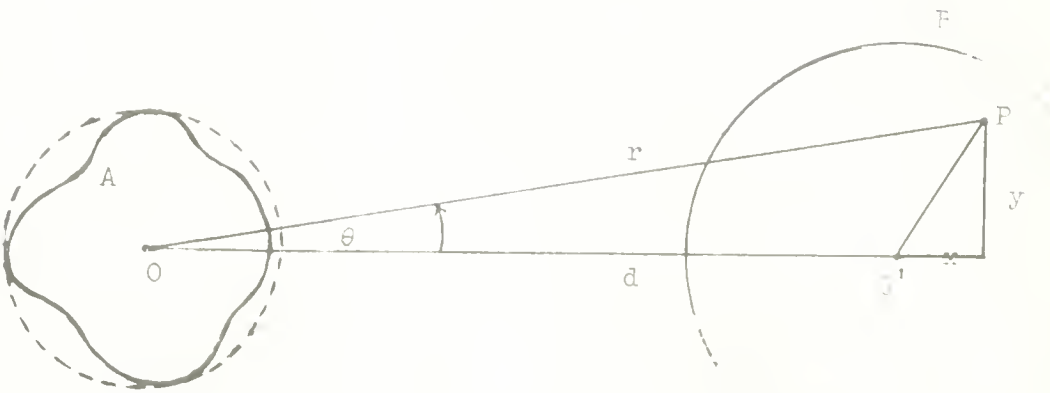



Figure 1

Let A be a fixed region of arbitrary size and let d be the distance from a point O at the center of the smallest circle containing A to the center O' of a circle B. Let the diameters of the circles A and B be small in comparison with d. Let P be a point in B other than O'. Let r and $\theta$ be the polar coordinates of p with origin at O, where $\theta$ is measured from the O-O' axis. Let x and y be cartesian coordinates of p with origin at O'. The principal result of this paper may then be stated as follows:

Theorem Consider any two-dimensional radiated field* represented by

* A radiated field is a solution of the reduced wave equation which satisfies the Sommerfeld radiation condition for an outgoing wave.

$$(1) \qquad u(r,\theta) = \int_{C_1} f(\beta) e^{ikr\cos(\theta-\beta)}\, d\beta$$

in the exterior of a bounded domain A with center O containing any combination of sources and scatterers where $C_1$ is the Sommerfeld contour for $H_0^{(1)}(kr)$ and $f(\beta)$ is analytic. Then $u(r,\theta)$ has an asymptotic expansion in any neighborhood $\beta$ of a point O' such that $\beta$ is disjoint with respect to the smallest circle containing A provided that $kd >> 1$ where d is the distance between O and O' (Figure 1). The expansion is given in terms of plane waves and their derivatives with respect to angle as follows:

$$(2) \qquad u \sim e^{ikd} \sum_{t=0}^{n} \frac{h^{(2t)}(0)}{(2t)!} \quad \frac{\Gamma(t+\frac{1}{2})}{(kd)^{t+1/2}} \quad \text{for any n,}$$

where

$$(2a) \qquad h^{(2t)}(0) = \sum_{p=0}^{2t} \sum_{\lambda=0}^{2t-p} \sum_{j=0}^{\lambda} \frac{C_p^{2t}\, C_j^{\lambda}}{\lambda!} (\beta^{\lambda}(0))^{(2t-p)} (\beta_s(0))^{(p)}\, f(0)^{[\lambda-j]}\, v(0)^{[j]}$$

and

$\Gamma(t)$ is the Gamma function, $C_j^{\lambda} = \frac{\lambda!}{j!(\lambda-j)!}$, $v(\beta) = e^{ik(x\cos\beta + y\sin\beta)}$

$$f(0)^{[\lambda-j]} = \left[\frac{d^{\lambda-j}}{d\beta^{\lambda-j}} f(\beta)\right]_{\beta=0}, \quad (\beta_s(0))^{(p)} = \left[\frac{d^{p+1}}{ds^{p+1}} \beta(s)\right]_{s=0},$$

$$= (\beta^{\lambda}(0))^{(2t-p)} \left[\frac{d^{2t-p}}{ds^{2t-p}} \beta^{\lambda}(s)\right]_{s=0}$$

$$\beta(s) = \arccos\,(1+is^2)\,, \quad \beta_s(0) = \left.\frac{d\beta}{ds}\right|_{s=0} .$$

Proof

Let

$$u(r,\theta) = \int_{C_1} e^{ikr\cos(\theta-\beta)}\, f(\beta)d\beta$$

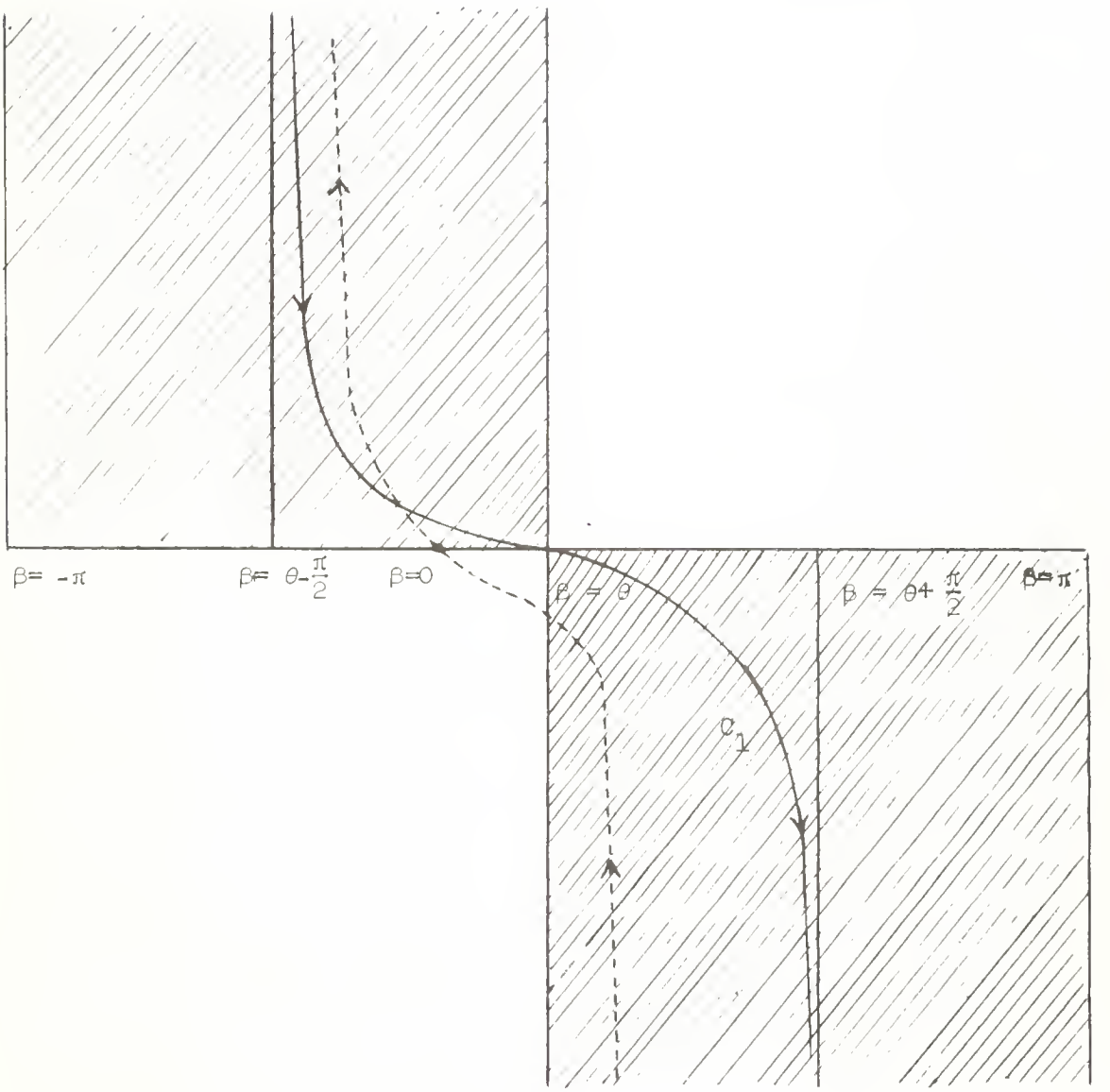
where $f(\beta)$ is analytic and $C_1$ is the Sommerfeld contour employed in the integral representation [5],[6] of $H_0^{(1)}(kr)$ (See figure 2). [The time dependence is assumed to be $e^{-i\omega t}$.] Expansion of $\cos(\theta-\beta)$ in the exponent of (1) in terms of x, y, and d (figure 1) yields

$$(3) \qquad u(r,\theta) = \int_{C_1} e^{ik(x\cos\beta+y\sin\beta)}\, f(\beta)e^{ikd\cos\beta}d\beta = \int_{C_1} g(\beta)e^{ikd\cos\beta}d\beta$$

where $g(\beta) = e^{ik(x\cos\beta+y\sin\beta)}f(\beta)$ .

It is clear that $e^{ikd\cos\beta}$ varies rapidly for large d and that the saddle point method is appropriate for evaluation of the integral. No difficulty is encountered in shifting the contour as d becomes large since the integrand is analytic. The transformation $\cos\beta = 1+is^2$ transforms $C_1$ into a steepest descent contour (figure 2) and (3) into

$$(4) \qquad u = e^{ikd}\int_{-\infty}^{\infty} g(\beta(s))\,\frac{d\beta}{ds}\,e^{-kds^2}\,ds = e^{ikd}\int_{-\infty}^{\infty} h\left[\beta(s)\right]e^{-kds^2}\,ds$$




Sommerfeld Contour ——— Saddle Contour of Steepest Descent ------

Figure 2

where $h\left[\beta(s)\right] \equiv ik(x \cos\beta(s)+y \sin\beta(s)) f(\beta(s)) \left(\frac{d\beta}{ds}\right)$ .

If $h\left[\beta(s)\right]$ is expanded in a Taylor series about the saddle point $s = 0$ and a finite number of these terms are retained, it is clear that

$$(5) \quad u \sim e^{ikd} \sum_{m=0}^{n} \left[\frac{h^{(m)}(0)}{m!} \int_{-\infty}^{\infty} s^m e^{-kds^2} ds\right] = e^{ikd} \sum_{m=0}^{n} \frac{h^{(m)}(0)}{m!} I_m$$

where $I_m = \int_{-\infty}^{\infty} s^m e^{-kds^2} ds$ and the superscribed parenthesis signifies differentiation with respect to $s$.

The function $h^{(m)}(s)$ can be represented in terms of derivatives of plane waves as follows:

$$h^{(m)} = (g\beta_s)^{(m)} = \sum_{p=0}^{m} C_p^m \, g^{(m-p)} \, \beta_s^{(p)}$$

by Leibnitz's Rule [7] .

A rule given by Goursat [8] for higher order derivatives of implicit functions yields

$$g^{(m-p)} = \sum_{\lambda=0}^{m-p} \sum_{\ell=0}^{\lambda} (-1)^{\ell} \frac{C_{\ell}^{\lambda}}{\lambda!} \beta^{\ell} (\beta^{\lambda-\ell})^{(m-p)} \, g^{[\lambda]}$$

where the square bracketed superscript signifies differentiation with respect to $\beta$. But $\beta(0) = 0$. Therefore,

$$g^{(m-p)}(0) = \sum_{\lambda=0}^{m-p} \frac{(\beta^{\lambda}(0))^{(m-p)}}{\lambda!} \, g^{[\lambda]}(0) .$$

A further application of Leibnitz's rule yields

$$g^{[\lambda]}(0) = \sum_{j=0}^{\lambda} C_j^{\lambda} \, f^{[\lambda-j]}(0) \, v^{[j]}(0) .$$

Therefore

$$h^{(m)}(0) = \sum_{p=0}^{m} \sum_{\lambda=0}^{m-p} \sum_{j=0}^{\lambda} \frac{C_p^m C_j^\lambda}{\lambda!} (\beta^\lambda(0))^{(m-p)} \beta_s^{(p)}(0) \; f_{(0)}^{[\lambda-j]} \; v_{(0)}^{[j]} .$$

Evaluation of $I_m$ results in

$$I_m = \begin{cases} 0 & m \text{ odd} \\ \dfrac{\Gamma(\frac{m+1}{2})}{(kd)^{\frac{m+1}{2}}} & m \text{ even} \end{cases} .$$

Let $m = 2t$

Then $I_{2t} = \begin{cases} 0 & m \text{ odd} \\ \dfrac{\Gamma(t+\frac{1}{2})}{(kd)^{\frac{m+1}{2}}} & m \text{ even} \end{cases}$ $\qquad t = 0,1,2, \ldots .$

and

$$u \sim e^{ikd} \sum_{t=0}^{n} \frac{h^{(2t)}(0)}{(2t)!} \frac{\Gamma(t+\frac{1}{2})}{(kd)^{t+1/2}} \tag{6}$$

where

$$h^{(2t)}(0) = \sum_{p=0}^{2t} \sum_{\lambda=0}^{2t-p} \sum_{j=0}^{\lambda} \frac{C_p^{2t} C_j^\lambda}{\lambda!} (\beta^\lambda(0))^{(2t-p)} (\beta_s(0))^{(p)} f_{(0)}^{[\lambda-j]} v_{(0)}^{[j]} . \tag{7}$$

It is convenient to note that

$$(8)\quad \beta_s(0)^{(p)} = \begin{cases} 0 & p \text{ odd} \\ 2^{\frac{p+1}{2}}\, e^{-i\left(\frac{p+1}{2}\right)\frac{\pi}{2}} \left(\frac{p!}{2^p\left(\frac{p}{2}\right)!}\right) = 2^{\nu+\frac{1}{2}}\, e^{-i\left(\nu+\frac{1}{2}\right)\frac{\pi}{2}} \left(\frac{(2\nu)!}{2^{2\nu}\nu!}\right) & p \text{ even} \end{cases}$$

where $p = 2\nu$, $\nu = 0,1,2 \ldots$ .

The symbol $\beta_s(0)^{(p)}$, evaluated in (8) means the $p^{th}$ derivative of $\frac{d\beta(s)}{ds}$ evaluated at $s = 0$.

It is clear, then, from (2) and (2a) that the coefficients in the form of derivatives of plane waves with respect to angle obtained previously [1] by the authors* were not accidental.

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* It should be pointed out here that the definition of the complex scattering amplitude of the far field $f(\beta)$ in this paper differs from that of the previous paper.[1] Let f denote the complex scattering amplitude of the far field in this paper and f denote the one in the previous paper. The relation between the two amplitudes is then

$$f = \sqrt{\frac{k}{2\pi}}\; e^{i\frac{\pi}{4}}\, f.$$

Application of the Theorem, and Discussion.

We shall now give an illustrative calculation in outline form. Comments on the nature of the work are also included. For an extended illustration see [4].

1. The function f(p) in the integral representation (1) for u is related to the scattering amplitude by the formulas.

$$u(r,\theta) \sim \sqrt{\frac{2}{\pi kr}}\ e^{ikr-\frac{\pi}{4}}\ [\hat{f}(\theta)]$$

$$\frac{1}{\pi}\,\hat{f}(\theta) \equiv f(\theta)\ .$$

2. $$\left.\frac{d\rho^{(s)}}{ds}\right|_{s=0} \equiv \sqrt{2}\quad e^{-\frac{i\pi}{4}}\quad .$$

3. To apply the formula (2a) to a radiated field it is sufficient to know $f(\theta)$ which determines the function u. Thus the radiated field need not be given in the form of our contour integral in order to apply our result. Instead, it suffices to know $f(\theta)$ in a small neighborhood of $\theta = 0$.

It is not necessary to know $f(\theta)$ in a large angular region. In this respect again our result differs from the more usual expansion theorem, which proceeds in inverse powers of r.

4. It is convenient to bear in mind that the functions $v^{[m]}(0)$ really contain the new coordinates x and y. For example

$$v(0) = e^{ik(x\cos\beta + y\sin\beta)}$$

$$v(0) = e^{ikx}$$

$$v^{[1]}(0) = ikye^{ikx} = \frac{dv(\beta)}{d\beta}\Big|_{\beta=0} ,$$

$$v^{[2]}(0) = ik\left[-x + iky^2\right]e^{ikx} = \frac{d^2v}{d\beta^2}\Big|_{\beta=0.} , \text{ etc.}$$

5. Consider for example the expansion of the function $H_o^{(1)}(kr)$. We have

$$H_o^{(1)}(kr) = H_o^{(1)}\left(k\sqrt{(x+d)^2+y^2}\right)$$

$$H_o^{(1)}(kr) \underset{r\to\infty}{\sim} \sqrt{\frac{2}{\pi kr}}\; e^{i(kr-\frac{\pi}{4})}\left[1+ O(\tfrac{1}{r})\right]$$

Hence $\hat{f}(\theta) \equiv 1$, $f(\theta) \equiv \frac{1}{\pi}$. Applying (2a) we find

$$H_o^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kd}}\; e^{i(kd-\frac{\pi}{4})}\left[e^{ikx} + \frac{e^{ikx} + 4ik\left[-x+iky\right]e^{ikx}}{8i(kd)} + \ldots\right]$$

$$= \sqrt{\frac{2}{\pi kd}}\; e^{i(kd-\frac{\pi}{4})}\left[v(0) + \frac{1}{8ikd}\left(v(0) + 4v^{[2]}(0)\right) + \ldots\right]$$

We shall now explain, in brief, how the expansion theorem may be employed in calculating the diffraction of an arbitrary radiating field.

In such a calculation the function $f(\theta)$ for the incident radiation will be known. The incident radiation will then be represented in the form (2a), in a coordinate system x,y fixed in the neighborhood of the scatterer.

In addition we must know the response amplitude $f_s(\theta,\theta_o)$ say, of the scatterer, in direction $\theta$, due to excitation by an incident plane wave $\exp ik\left[x\cos\theta_o + y\sin\theta_o\right] = v(\theta_o)$. Since $\theta_o$ is a parameter, we can then

calculate the response to any of the functions $v^{[\lambda]}(0)$ in terms of which the incident radiation will have been expressed, since the response to a derivative of a plane wave is the derivative of the response to the plane wave, each term of the response to this incident radiation will have a coefficient identical to the one which appears in the expansion theorem (a linear combination of derivatives with respect to angle of incidence) operating upon the response to a plane wave of a given angle of incidence. For a detailed illustration cf[4] and [1].

REFERENCES

[1] Zitron, N. R. and Karp, S. N., Higher order approximations in multiple scattering. I. Two-dimensional scalar case; J. Mathematical Phys., 2, 3, 394-402 (1961); See also N.Y.U. Inst. Math. Sci., Div. of E. M. Research, Research Report No. EM-126, March, 1959.

[2] Karp, S. N. and Zitron, N. R., A new expansion theorem for radiated fields; Bull. Amer. Phys. Soc., Series II, 4, No. 4, 243 (1959).

[3] Borowitz, S., Basic mathematical investigations in electromagnetic wave theory; N.Y.U. Inst. of Math. Sci., Div. of E.M. Res., Final Report under Contract No. AF19(604)1717, March, 1959.

[4] Zitron, N. and Davis, J., A numerical investigation of scattering of radiation of an asymmetric source by circular cylinder, U. of Wisconsin, Math. Res. Center, MRC Tech. Summary Report No. 365, March, 1963.

[5] Stratton, J. A., Electromagnetic Theory, New York, McGraw-Hill 1941, p. 365.

[6] Hartman, P. and Wilcox, C., On solutions of the Helmholtz equation in exterior domains; Math. Zeitschr. 75, 228-255 (1961).

[7] Courant, R., Differential and Integral Calculus, London and Glasgow, Blackie and Sons Ltd. 1937, Vol. I, p. 202.

[8] Goursat-Hedrick, Mathematical Analysis, Boston, Ginn and Co. 1904, Vol. I, p. 32.

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